

MIN-OO CONJECTURE FOR FULLY NONLINEAR CONFORMALLY INVARIANT EQUATIONS

EZEQUIEL BARBOSA, MARCOS P. CAVALCANTE, AND JOSÉ M. ESPINAR

ABSTRACT. In this paper we show rigidity results for super-solutions to fully nonlinear elliptic conformally invariant equations on subdomains of the standard n -sphere \mathbb{S}^n under suitable conditions along the boundary.

This proves rigidity for compact connected locally conformally flat manifolds (M, g) with boundary such that the eigenvalues of the Schouten tensor satisfy a fully nonlinear elliptic inequality and whose boundary is isometric to a geodesic sphere $\partial D(r)$, where $D(r)$ denotes a geodesic ball of radius $r \in (0, \pi/2]$ in \mathbb{S}^n , and totally umbilic with mean curvature bounded below by the mean curvature of this geodesic sphere. Under the above conditions, (M, g) must be isometric to the closed geodesic ball $\bar{D}(r)$. In particular, we recover the solution by F.M. Spiegel [23] to the Min-Oo conjecture for locally conformally flat manifolds.

As a side product, our methods in dimension 2 provide a new proof to Toponogov's Theorem [24]. In fact, we can extend it (see Theorem 6.2). Roughly speaking, Toponogov's Theorem is equivalent to a rigidity theorem for spherical caps in the Hyperbolic three-space \mathbb{H}^3 .

1. INTRODUCTION

In 1995, Min-Oo [17], inspired by the work of Schoen and Yau [19, 20] on the Positive Mass Theorem, conjectured that if (M^n, g) is a compact Riemannian manifold with boundary such that the scalar curvature of M is at least $n(n-1)$ and whose boundary ∂M is totally geodesic and isometric to the standard sphere, then M is isometric to the closed hemisphere $\bar{\mathbb{S}}_+^n$ equipped with the standard round metric. This conjecture were verified in many particular cases, but it is false in general. In fact, Brendle, Marques and Neves in [4] constructed some families of metrics on $\bar{\mathbb{S}}_+^n$, $n \geq 3$, that are counter-examples to this conjecture.

One important case where Min-Oo's conjecture is true, proved by Hang and Wang in [10], is when we consider metrics in the conformal class to the standard metric on the hemisphere:

Theorem 1.1 (Hang-Wang [10]). *Let $g = e^{2\rho}g_0$ be a C^2 metric on the unit closed hemisphere $\bar{\mathbb{S}}_+^n$, where g_0 denotes the standard round metric. Assume that*

- (a) $R_g \geq n(n-1)$, and
- (b) *the boundary is totally geodesic and isometric to the standard \mathbb{S}^{n-1} .*

Then g is isometric to g_0 .

2010 *Mathematics Subject Classification.* Primary 53C21, 53C24; Secondary 58J05.

Key words and phrases. Scalar Curvature, Min-Oo's Conjecture, Conformally Invariant Equations.

The authors were partially supported by CNPq/Brazil.

Recently, Spiegel [23] showed a scalar curvature rigidity theorem for locally conformally flat manifolds with boundary in the spirit of Min-Oo's conjecture which is an extension of Hang-Wang's Theorem. To be more precise, let $p \in \mathbb{S}^n$, $0 < r \leq \frac{\pi}{2}$ and

$$D(p, r) := \{x \in \mathbb{S}^n : d_{g_0}(x, p) < r\}$$

be the geodesic ball of radius r centered at p in \mathbb{S}^n . Let $H_r = \cot(r)$ be the mean curvature of the boundary $\partial D(p, r)$, measured with respect to the inward orientation. Note that $\partial D(p, r)$ is isometric to a sphere of radius $\sin(r)$.

Theorem 1.2 (Spiegel [23]). *Let (M^n, g) , $n \geq 3$, be a compact connected locally conformally flat Riemannian manifold with boundary. Assume that*

- (a) $R_g \geq n(n-1)$, and
- (b) *the boundary ∂M is umbilic with mean curvature $H_g \geq H_r$ and isometric to $\partial D(p, r)$, $0 < r \leq \pi/2$. Here, the mean curvature is measured with respect to the inward orientation.*

Then (M, g) is isometric to $\overline{D(p, r)}$ with the standard metric.

Remark 1.3. Spiegel also proved that the assumption on the mean curvature in the theorem above can be dropped provided M is simply-connected and $r = \frac{\pi}{2}$. See Remark 1.3 in [23]. Therefore, Theorem 1.2 is an extension of Theorem 1.1.

Theorem 1.2 is sharp in r in the sense that one can construct counterexamples on $\overline{D(p, r)}$ for $\pi/2 < r < \pi$ (cf. [10]).

We are interested in the Min-Oo's conjecture for compact connected locally conformally flat Riemannian manifolds (M^n, g) satisfying a more general curvature condition. It is well known that the scalar curvature is, up to a constant, the sum of the eigenvalues of the Schouten tensor Sch_g . In fact, let $\lambda(p) = (\lambda_1(p), \dots, \lambda_n(p))$ denote its eigenvalues, then

$$(1.1) \quad \text{Trace}(g^{-1}\text{Sch}_g) = \lambda_1(p) + \dots + \lambda_n(p) = \frac{R(g)}{2(n-1)}.$$

It is natural to ask if the Min-Oo's conjecture holds when one considers a more general function on the eigenvalues of the Schouten tensor instead of the scalar curvature. Of special interest is when we consider $\sigma_k(\lambda(p))$, the k -th elementary symmetric polynomial of the eigenvalues $\lambda_1(p), \dots, \lambda_n(p)$.

In order to establish properly our main result, we need to define the type of curvature function for the eigenvalues of the Schouten tensor that we will consider. First, let us recall the notion of elliptic data originally introduced by Caffarelli, Nirenberg and Spruck [5]; we use the theory developed by Li and Li for conformal equations (cf. [14, 15]). Consider the convex cones

$$\begin{aligned} \Gamma_n &= \{x \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}, \\ \Gamma_1 &= \{x \in \mathbb{R}^n : x_1 + \dots + x_n > 0\}. \end{aligned}$$

Let $\Gamma \subset \mathbb{R}^n$ be a symmetric open convex cone and $f \in C^1(\Gamma) \cap C^0(\overline{\Gamma})$. We say that (f, Γ) is an *elliptic data* if the pair (f, Γ) satisfies

- (1) $\Gamma_n \subset \Gamma \subset \Gamma_1$,
- (2) f is symmetric,
- (3) $f > 0$ in Γ ,
- (4) $f|_{\partial\Gamma} = 0$,
- (5) f is homogeneous of degree 1,

- (6) $\nabla f(x) \in \Gamma_n$ for all $x \in \Gamma$,
- (7) $f(1, \dots, 1) = 2$.

Let (M, g) be a Riemannian manifold. Then, given an elliptic data (f, Γ) we say that g is a *supersolution* to (f, Γ) if

$$f(\lambda_g(p)) \geq 1, \lambda_g(p) \in \Gamma \text{ for all } p \in M,$$

where $\lambda_g(p) = (\lambda_1(p), \dots, \lambda_n(p))$ is composed by the eigenvalues of the Schouten tensor of g at $p \in M$.

It is well-known that the Schouten tensor of the standard n -sphere is $\text{Sch}_{g_0} = \frac{1}{2}g_0$, then, condition (7) above says that we are normalizing the functional f to be 1 when considering the Schouten tensor of the standard sphere, i.e.,

$$f(1/2, \dots, 1/2) = 2^{-1}f(1, \dots, 1) = 1,$$

where we have used that f is homogeneous of degree one.

In this paper, we prove that the Min-Oo's conjecture holds for super-solutions to elliptic data in locally conformally flat manifolds. Namely, we prove the following result.

Theorem A. *Let (M^n, g) be a compact connected locally conformally flat Riemannian manifold with boundary ∂M . Let (f, Γ) be an elliptic data and assume that g is a supersolution to (f, Γ) in M , i.e.,*

$$f(\lambda_g(p)) \geq 1, \lambda_g(p) \in \Gamma \text{ for all } p \in M.$$

Assume that ∂M is umbilical with mean curvature $H_g \geq H_r$ and isometric to $\partial D(p, r)$, $0 < r \leq \pi/2$. Then (M, g) is isometric to $\overline{D(p, r)}$ with the standard metric.

In addition, if M is simply-connected and ∂M is totally geodesic and isometric to \mathbb{S}^{n-1} , we do not need to assume that $H_g \geq 0$.

Our approach relies in a geometric method developed by the third author, Gálvez and Mira in [8] and further developments contained in [1, 2, 3, 6, 7], where conformal metrics on spherical domains are represented by hypersurfaces in the hyperbolic space. In order to reduce our problem on locally conformally flat manifolds to conformal metrics on subdomains of the sphere, we use results contained in the work of Spiegel [23] and Li and Nguyen [16] based on the deep theory by Schoen and Yau [21] on the developing map of a locally conformally flat manifold. Hence, combining these results, we show that Theorem A is equivalent to a rigidity result for horospherically concave hypersurfaces with boundary in the Hyperbolic space \mathbb{H}^{n+1} . In particular, in dimension $n = 2$, these methods provide a new proof to Toponogov's Theorem [24] and, in fact, we can extend it.

2. PRELIMINARIES

We will establish in this section the necessary tools we will use along this paper.

2.1. Representation formula and Regularity. Here we recover the hypersurface interpretation of conformal metrics on the sphere developed in [2, 8]. Let us denote by \mathbb{L}^{n+2} the Minkowski spacetime, that is, the vector space \mathbb{R}^{n+2} endowed with the Minkowski spacetime metric \langle, \rangle given by

$$\langle \bar{x}, \bar{x} \rangle = -x_0^2 + \sum_{i=1}^{n+1} x_i^2,$$

where $\bar{x} \equiv (x_0, x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+2}$.

Then hyperbolic space, de Sitter spacetime and positive null cone are given, respectively, by the hyperquadrics

$$\begin{aligned}\mathbb{H}^{n+1} &= \{\bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = -1, x_0 > 0\} \\ d\mathbb{S}_1^{n+1} &= \{\bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = 1\} \\ \mathbb{N}_+^{n+1} &= \{\bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = 0, x_0 > 0\}.\end{aligned}$$

Let $\phi : M^n \rightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be an isometric immersion of an oriented hypersurface, with orientation $\eta : M^n \rightarrow d\mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$. We define the associated light cone map as

$$\psi := \phi - \eta : M^n \rightarrow \mathbb{N}_+^{n+1} \subset \mathbb{L}^{n+2}.$$

If we write $\psi = (\psi_0, \dots, \psi_{n+1})$, consider the map G (the hyperbolic Gauss map) given by:

$$G = \frac{1}{\psi_0}(\psi_1, \dots, \psi_{n+1}) : M \rightarrow \mathbb{S}^n,$$

Hence, if we label $e^\rho := \psi_0$ (the hyperbolic support function), we get

$$\psi = e^\rho(1, G) \in \mathbb{L}^{n+2}.$$

Set $\Sigma := \phi(M^n) \subset \mathbb{H}^{n+1}$ with orientation η . We say that Σ is horospherically concave if Σ lies (locally) around any point $p \in \Sigma$ strictly in the concave side of the tangent horosphere at p and its normal points into the concave side of the tangent horosphere (cf. Figure 1).

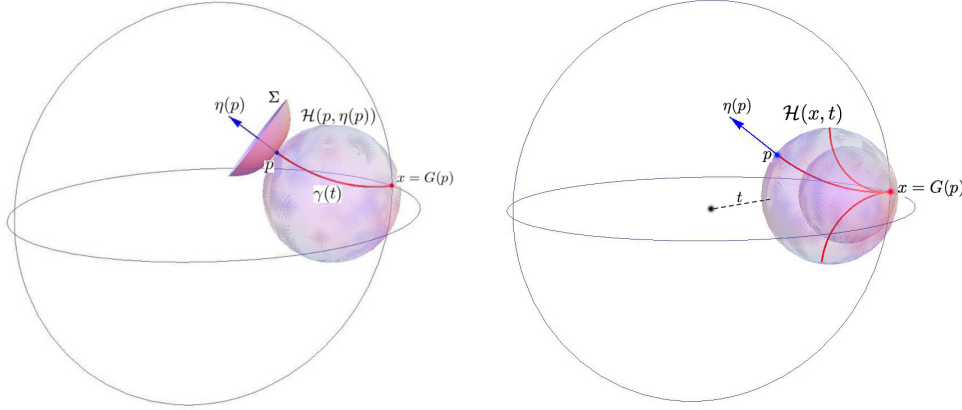


FIGURE 1. **Left:** A (local) horospherically concave hypersurface $\Sigma = \phi(M)$ with canonical orientation η . **Right:** A horosphere with the outward orientation has constant hyperbolic Gauss map.

Theorem 2.1 ([8]). *Let $\phi : \Omega \subset \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ be an oriented piece of horospherically concave hypersurface with orientation $\eta : \Omega \rightarrow d\mathbb{S}_+^{n+1}$ and hyperbolic Gauss map $G(x) = x$. Then*

$$(2.1) \quad \phi(x) = \frac{e^\rho}{2}(1 + e^{-2\rho}(1 + \|\nabla \rho\|^2))(1, x) + e^{-\rho}(0, -x + \nabla \rho),$$

and its orientation is given by

$$(2.2) \quad \eta(x) = \phi(x) - e^\rho(1, x).$$

Moreover, the eigenvalues λ_i of the Schouten tensor of $g = e^{2\rho}g_0$ and the principal curvatures k_i of ϕ are related by

$$\lambda_i = \frac{1}{2} - \frac{1}{1 + k_i}.$$

Conversely, given a conformal metric $g = e^{2\rho}g_0$ defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that the eigenvalues of its Schouten tensor are all less than $1/2$, then the map ϕ given by (2.1) defines an immersed, horospherically concave hypersurface in \mathbb{H}^{n+1} with orientation (2.2) whose hyperbolic Gauss map is $G(x) = x$ for $x \in \Omega$.

Here, the connection ∇ and the norm $\|\cdot\|$ are with respect to the standard metric g_0 on \mathbb{S}^n .

Let $\Omega \subset \mathbb{S}^n$ be a relatively compact domain with smooth boundary. Given $\rho \in C^2(\bar{\Omega})$, the above representation formula says that ϕ and η are C^1 maps and $\Sigma := \phi(\bar{\Omega}) \subset \mathbb{H}^{n+1}$ is a compact hypersurface with boundary $\partial\Sigma = \phi(\partial\Omega)$ whose tangent plane varies C^1 . Moreover, the corresponding conformal metric $g = e^{2\rho}g_0$ on Ω is the horospherical metric associated to Σ . Observe that, since $\rho \in C^2(\bar{\Omega})$, the eigenvalues of the Schouten tensor associated to $g = e^{2\rho}g_0$ are continuous in Ω and hence there exists $t > 0$ so that the eigenvalues of the Schouten tensor associated to $g_t = e^{2(\rho+t)}g_0$ are less than $1/2$.

In the Poincaré ball model of \mathbb{H}^{n+1} , the representation formula (cf. [1]) is given by

$$\varphi_t(x) = \frac{1 - e^{-2\rho_t(x)} + \|\nabla e^{-\rho_t}(x)\|^2}{(1 + e^{-\rho_t(x)})^2 + \|\nabla e^{-\rho_t}(x)\|^2} x - \frac{1}{(1 + e^{-\rho_t(x)})^2 + \|\nabla e^{-\rho_t}(x)\|^2} \nabla(e^{-2\rho_t})(x).$$

Set $\epsilon = e^{-t}$, then

$$f(x, \epsilon) := -\frac{2(e^{\rho(x)} + \epsilon)}{(e^{\rho(x)} + \epsilon)^2 + \epsilon^2 \|\nabla \rho(x)\|^2}$$

and

$$g(x, \epsilon) = \frac{2\epsilon}{(e^{\rho(x)} + \epsilon)^2 + \epsilon^2 \|\nabla \rho(x)\|^2}$$

are in $C^1(\bar{\Omega} \times [0, +\infty))$ and they are smooth in ϵ , moreover, the vector field $\nabla \rho$ is C^1 in $\bar{\Omega}$, since $\rho \in C^2(\bar{\Omega})$. Thus,

$$\varphi_\epsilon(x) = x + \epsilon(f(x, \epsilon)x + g(x, \epsilon)\nabla \rho(x)) \in \mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$$

belongs to $C^1(\bar{\Omega})$, in particular, the vector field

$$Y(x, \epsilon) := f(x, \epsilon)x + g(x, \epsilon)\nabla \rho(x) \in C^1(\bar{\Omega} \times [0, +\infty)).$$

Let $\tilde{Y} : \mathbb{S}^n \times [0, +\infty) \rightarrow \mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$ be the Lipschitz extension of Y so that $\tilde{Y}|_{\bar{\Omega} \times [0, +\infty)} = Y$. Therefore, the corresponding extension map

$$\tilde{\varphi} : \mathbb{S}^n \times [0, +\infty) \rightarrow \mathbb{R}^{n+2}$$

is Lipschitz in x and smooth in ϵ so that $\tilde{\varphi}(x, \epsilon) = \varphi_\epsilon(x)$ for all $(x, \epsilon) \in \bar{\Omega} \times (0, +\infty)$ satisfying $\tilde{\varphi}(x, 0) = x$, i.e., $\tilde{\varphi}_0(\cdot) = \tilde{\varphi}(\cdot, 0)$ is the identity map, which is an embedding of the sphere \mathbb{S}^n into \mathbb{R}^{n+1} . Since $\tilde{\varphi}_\epsilon : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ is a Lipschitz deformation of an embedding, from [9], there exists $\epsilon_0 > 0$ so that $\tilde{\varphi}_\epsilon : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$

is an embedding for all $\epsilon \in [0, \epsilon_0)$. Thus, summarizing all we have done in this Subsection, we obtain:

Lemma 2.2 ([1, 2, 8]). *Let $\Omega \subset \mathbb{S}^n$ be a relatively compact domain with smooth boundary and $\rho \in C^2(\overline{\Omega})$. Then, there exists $t > 0$ so that the horospherically concave hypersurface $\phi_t : \overline{\Omega} \rightarrow \mathbb{H}^{n+1}$ given by (2.1) is a compact embedded hypersurface $\Sigma_t = \phi_t(\Omega)$ with boundary $\partial\Sigma_t = \phi_t(\partial\Omega)$. Moreover, the eigenvalues of its associated horospherical metric $g_t := e^{2(\rho+t)}g_0$ are less than $1/2$.*

It is important to recall the connection between isometries of the hyperbolic space $\text{Iso}(\mathbb{H}^{n+1})$ and conformal diffeomorphisms of the sphere $\text{Conf}(\mathbb{S}^n)$. It is well-known that each isometry $T \in \text{Iso}(\mathbb{H}^{n+1})$ induces a unique conformal diffeomorphism $\Phi \in \text{Conf}(\mathbb{S}^n)$.

Let $T \in \text{Iso}(\mathbb{H}^{n+1})$ be an isometry and $\Phi \in \text{Conf}(\mathbb{S}^n)$ be the unique conformal diffeomorphism associated to T . Then, given a horospherically concave hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ with horospherical metric g , one can see that (cf. [7]) the horospherical metric \tilde{g} associated to $\tilde{\Sigma} = T(\Sigma)$ is given by $\tilde{g} = \Phi^*g$. Vice versa, given a conformal metric g on a subdomain of the sphere with associated hypersurface Σ , given by the representation formula under the appropriated conditions, the associated horospherically concave hypersurface $\tilde{\Sigma}$ associated to the conformal metric $\tilde{g} = \Phi^*g$ is given by $\tilde{\Sigma} = T(\Sigma)$.

2.2. Locally conformally flat metrics and developing map. Let (M^n, g) , $n \geq 3$, be a Riemannian manifold with a C^k -metric g . We say that (M, g) is locally conformally flat if for every point $p \in M$ there exist a neighborhood U of p and $\varphi \in C^k(U)$ such that the metric $e^{2\varphi}g$ is flat on U . An immersion $\Psi : (M, g) \rightarrow (N, h)$ is a conformal immersion if we can write $\Psi^*h = e^{2\varphi}g$ for some function φ .

If (M, g) is a locally conformally flat manifold it is well known that there exists a conformal map $\Psi : M \rightarrow \mathbb{S}^n$, called the *developing map* which is unique up to conformal transformations of \mathbb{S}^n . When M is compact and simply-connected with umbilical boundary, Spiegel [23] proved that the developing map can be taken as a diffeomorphism over the hemisphere $\overline{\mathbb{S}^n}_+$.

If M is not simply-connected, we can pass to the universal covering \tilde{M} to obtain a developing map $\Psi : \tilde{M} \rightarrow \mathbb{S}^n$ which is, under some assumptions, injective. In fact, Spiegel [23] (see also [16]) showed the following theorem:

Theorem 2.3. *Let (M, g) be a compact connected locally conformally flat manifold with boundary. Assume that M has positive scalar curvature and that ∂M is umbilic and simply-connected with non-negative mean curvature. Let $\Pi : \tilde{M} \rightarrow M$ be the universal covering. Then there exists an injective conformal map $\Psi : \tilde{M} \rightarrow \mathbb{S}^n$ which is a conformal diffeomorphism onto its image. The image is of the form*

$$\Omega = \Omega(\epsilon_i, p_i, \Lambda) := \mathbb{S}^n \setminus \left(\bigcup_i D(p_i, \epsilon_i) \cup \Lambda \right),$$

where the $D(p_i, \epsilon_i)$ are geodesic balls in \mathbb{S}^n centered at p_i of radius ϵ_i with disjoint closures and Λ is the so-called limit set, a closed subset of Hausdorff dimension at most $\frac{n-2}{2}$.

Note that, since we are assuming $\lambda_g(p) \in \Gamma$, for all $p \in M$, and $\Gamma \subset \Gamma_1$, hence we have that $R_g > 0$. Therefore, under the conditions of Theorem A, we can apply Theorem 2.3.

3. THE CASE OF THE HEMISPHERE

We begin by considering the baby case, say conformal metrics on the hemisphere. This case will enlighten the geometric ideas contained in the proof.

Theorem 3.1. *Let (f, Γ) be an elliptic data and let $g = e^{2\rho}g_0$, $\rho \in C^2(\overline{\mathbb{S}_+^n})$, $n \geq 3$, be a supersolution to (f, Γ) on the closed hemisphere $\overline{\mathbb{S}_+^n}$, i.e.,*

$$f(\lambda(p)) \geq 1, \quad \lambda_g(p) \in \Gamma \text{ for all } p \in \mathbb{S}_+^n.$$

*Assume that the boundary $\partial\mathbb{S}_+^n$ with respect to g is isometric to $\partial\mathbb{S}_+^n$. Then $g = \Phi^*g_0$, where $\Phi \in \text{Conf}(\mathbb{S}^n)$ preserving $\overline{\mathbb{S}_+^n}$.*

Proof. First, $\partial\mathbb{S}_+^n$ is isometric to \mathbb{S}^{n-1} implies that $g|_{\partial\mathbb{S}_+^n}$ is isometric to \mathbb{S}^{n-1} . Hence, by Obata's Theorem, there exists a conformal diffeomorphism $\tilde{\Phi} \in \text{Conf}(\mathbb{S}^{n-1})$ so that $g|_{\partial\mathbb{S}_+^n} = \tilde{\Phi}^*g_0|_{\partial\mathbb{S}_+^n}$ along $\partial\mathbb{S}_+^n$. Observe that $\tilde{\Phi}$ can be extended to a conformal diffeomorphism $\Phi \in \text{Conf}(\mathbb{S}^n)$ so that $\Phi(\mathbb{S}_+^n) = \mathbb{S}_+^n$ and $\Phi|_{\partial\mathbb{S}_+^n} = \tilde{\Phi}$. Hence, up to the conformal diffeomorphism Φ , we can assume that $g = g_0$ along $\partial\mathbb{S}_+^n$. In other words,

$$(3.1) \quad \rho = 0 \text{ on } \partial\mathbb{S}_+^n.$$

Moreover, since $\partial\mathbb{S}_+^n$ is totally geodesic with respect to g_0 and g is conformal to g_0 , $\partial\mathbb{S}_+^n$ is totally umbilical with respect to g , in particular, the mean curvature along $\partial\mathbb{S}_+^n$ with respect to g is given by

$$(3.2) \quad H_g := -e^{-\rho} \frac{\partial \rho}{\partial \nu} = -\frac{\partial \rho}{\partial \nu} \text{ on } \partial\mathbb{S}_+^n,$$

where $\nu = e_{n+1}$ is the inward normal along $\partial\mathbb{S}_+^n$.

Let $P \subset \mathbb{H}^{n+1}$ be the totally geodesic hyperplane whose boundary at infinity is the equator of the upper hemisphere, i.e., $\partial_\infty P = \partial\mathbb{S}_+^n$. Denote by P^+ (resp. P^-) the connected component of $\mathbb{H}^{n+1} \setminus P$ that contains the north pole (resp. south pole) at its boundary at infinity. Also, denote by $P(b)$, $b \in \mathbb{R}$, the equidistant to P at distance b . Note that $P(b) \subset P^+$ when $b > 0$ and $P(b) \subset P^-$ when $b < 0$. We define $P(b)^+$ (resp. $P(b)^-$) as the connected component of $\mathbb{H}^{n+1} \setminus P(b)$ containing the north pole (resp. south pole) in its boundary at infinity. Clearly, $\partial_\infty P(b) = \partial_\infty P = \partial\mathbb{S}_+^n$ for all $b \in \mathbb{R}$.

Now, we fix $t > 0$ as in Lemma 2.2 such that the eigenvalues of the Schouten tensor of $g_t = e^{2(\rho+t)}g_0$ satisfy $\lambda_i^t(x) < 1/2$ for all $x \in \overline{\mathbb{S}_+^n}$ and the compact horospherically concave hypersurface with boundary $\Sigma_t = \phi_t(\mathbb{S}_+^n) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ given by the representation formula (2.1) associated to $\rho_t = \rho + t$ is embedded. Given $p \in \mathbb{H}^{n+1}$ we denote by $d_{\mathbb{H}^{n+1}}(p, P)$ the signed distance to P , that is, it is positive if $p \in P^+$ and negative if $p \in P^-$. Then, taking $t > 0$ big enough in Lemma 2.2 we can assume that Σ_t is above $P(m)$, i.e., $\Sigma_t \subset \overline{P(m)^+}$, where $m = \min \{d_{\mathbb{H}^{n+1}}(p, P) : p \in \partial\Sigma_t\}$. In fact, one can check (cf. [1, Section 2.4] for details) that $m = \min \{\text{arc sinh}(-e^{-t}H_g(x)) : x \in \partial\mathbb{S}_+^n\}$.

Observe that (3.1) implies

$$(3.3) \quad \rho_t = t \text{ and } \frac{\partial \rho_t}{\partial \nu} = \frac{\partial \rho}{\partial \nu} \text{ on } \partial\mathbb{S}_+^n.$$

We claim:

Claim A: Let $\gamma : \mathbb{R} \rightarrow \mathbb{H}^{n+1}$ be the complete geodesic (parametrized by arc-length) joining the south and north poles. Let \mathcal{C}_t be the solid cylinder in \mathbb{H}^{n+1} of axis γ and radius t . Then, $\partial\Sigma_t$ lies outside the interior of \mathcal{C}_t , and $\partial\Sigma_t \cap \mathcal{C}_t \subset P$. Moreover, if $\partial\Sigma_t \cap \mathcal{C}_t \neq \emptyset$ then at such points Σ_t is orthogonal to P .

Proof of Claim A. Note that, since $x \in \partial\mathbb{S}_+^n$, $\phi_t(x) \in \mathcal{H}(x, t)$, where $\mathcal{H}(x, t)$ is the horosphere whose point at infinity is x and signed distance to the origin is $t > 0$ (see [2]). It proves the first part of the claim (cf. Figure 2).

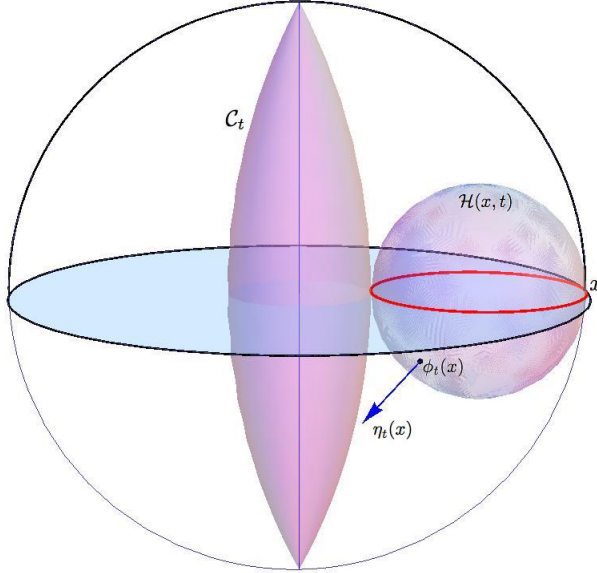


FIGURE 2. For any $x \in \partial\mathbb{S}_+^n$ we have $\phi_t(x) \in \mathcal{H}(x, t)$, since $\rho_t(x) = t$, and the canonical orientation $\eta_t(x)$ agrees with the outward orientation of $\mathcal{H}(x, t)$ at $\phi_t(x)$, since Σ_t is horospherically concave.

To finish the proof, we must check that at a point where $\frac{\partial\rho}{\partial\nu}(x) = 0$ we get that Σ_t is orthogonal to P . The unit normal along Σ_t is given by

$$\eta_t(x) = \frac{e^{-\rho-t}}{2} (\|\nabla\rho\|^2 - 1 + e^{\rho+t})(1, x) + e^{-\rho-t}(0, -x + \nabla\rho)$$

and the normal along P is given by $n(p) = (0, e_{n+1})$ for all $p \in P$. Hence, we have

$$\langle \eta_t(x), n(\phi(x)) \rangle = 0,$$

that is, Σ_t is orthogonal to P at x . □

Let $(1, \mathbf{0}) := (1, 0, \dots, 0) \in \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be the origin in the hyperboloid model (note that such point corresponds to the actual origin in the Poincaré ball model). Denote by $S_t \subset \mathbb{H}^{n+1}$ the geodesic sphere centered at the origin $(1, \mathbf{0})$ of radius t .

It is easy to see that its horospherical metric is given by $\tilde{g}_t = e^{2t}g_0$ (cf. [7]). Consider the half-sphere $S_t^+ = S_t \cap \overline{P}^+$ and observe that S_t^+ is orthogonal to P along the boundary ∂S_t^+ (cf. Figure 3).

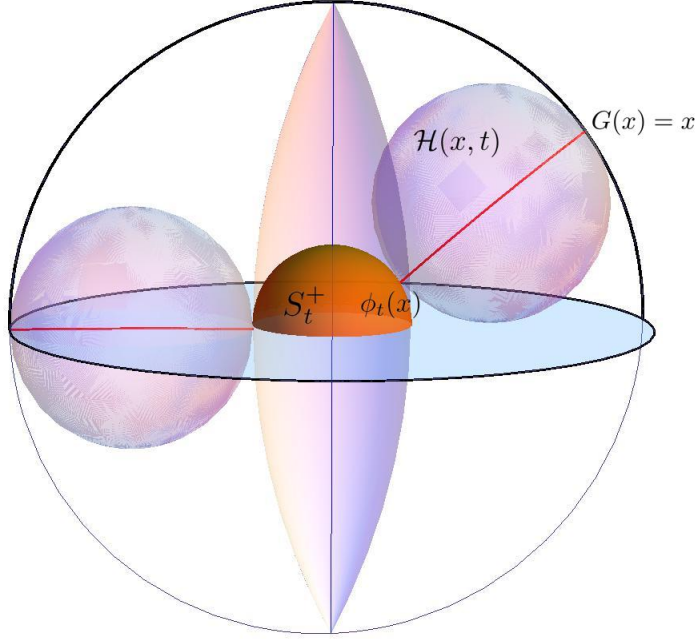


FIGURE 3. The half-sphere $S_t^+ \setminus \partial S_t^+$ is inside the cylinder \mathcal{C}_t and only touches it along the boundary ∂S_t^+ . Moreover, observe that the hyperbolic Gauss map is the identity and the hyperbolic support function is the constant t function.

Let $T_s : \mathbb{H}^{n+1} \rightarrow \mathbb{H}^{n+1}$ be the hyperbolic translation at distance s along γ so that $T_s((1, \mathbf{0})) = \gamma(s)$, an isometry of \mathbb{H}^{n+1} . It is clear that $T_s(S_t^+ \setminus \partial S_t^+) \cap \partial \Sigma_t = \emptyset$, for all $s \in \mathbb{R}$ by Claim A.

Let $\Phi_s \in \text{Conf}(\mathbb{S}^n)$ be the unique conformal diffeomorphism associated to T_s . Set $S_{t,s} := T_s(S_t)$ for all $s \in \mathbb{R}$, then the horospherical metric associated to $S_{t,s}$ is given by $\tilde{g}_{t,s} = e^{2t} \Phi_s^* g_0$ in \mathbb{S}^n and denote by $\tilde{\rho}_{t,s} \in C^\infty(\mathbb{S}^n)$ the horospherical support function associated to $S_{t,s}$, i.e., $\tilde{g}_{t,s} = e^{2\tilde{\rho}_{t,s}} g_0$ (cf. Figure 4 Left). Let $\hat{g}_{t,s}$ be the restriction of $\tilde{g}_{t,s}$ to $\overline{\mathbb{S}_+^n}$, i.e., $\tilde{g}_{t,s}|_{\mathbb{S}_+^n} = \hat{g}_{t,s}$, and $\hat{\rho}_{t,s}$ the restriction of $\tilde{\rho}_{t,s}$ to $\overline{\mathbb{S}_+^n}$ (cf. Figure 4 Right).

Consider $\bar{s} \in \mathbb{R}$ so that $S_{t,s}^+ \cap \Sigma_t = \emptyset$ for all $s < \bar{s}$. Increasing s from \bar{s} to $+\infty$, we must find a first instant s_0 so that $S_{t,s_0}^+ \cap \Sigma_t \neq \emptyset$ tangentially. If S_{t,s_0}^+ does not coincide with Σ_t identically, such tangential point must be either at an interior point of Σ_t or at a boundary point of $\partial \Sigma_t$. In the latter case we must necessarily have $s_0 = 0$ by the second part of Claim A.

Claim B: $\rho_t \geq \hat{\rho}_{t,s_0}$ on $\overline{\mathbb{S}_+^n}$.

Proof of Claim B. From Claim A we have that $\mathcal{H}(x, r)$ either does not touch S_{t,s_0} or does touch at a tangential point, for all $x \in \partial \mathbb{S}_+^n$ and all $r \geq t$. This says that $\rho_t \geq \hat{\rho}_{t,s_0}$ on $\partial \mathbb{S}_+^n$ because Σ_t is horospherically concave. Now, let us prove that $\rho_t \geq \hat{\rho}_{t,s_0}$ on \mathbb{S}_+^n . Assume there exists $x \in \mathbb{S}_+^n$ so that $\rho_t(x) < \hat{\rho}_{t,s_0}(x)$. Then, as pointed out above, the horosphere $\mathcal{H}(x, \hat{\rho}_{t,s_0}(x))$ does not touch Σ_t and touch at one point $q \in S_{t,s_0}$. Observe that $\mathcal{H}(x, \hat{\rho}_{t,s_0}(x) - \delta)$ does not touch Σ_t for any

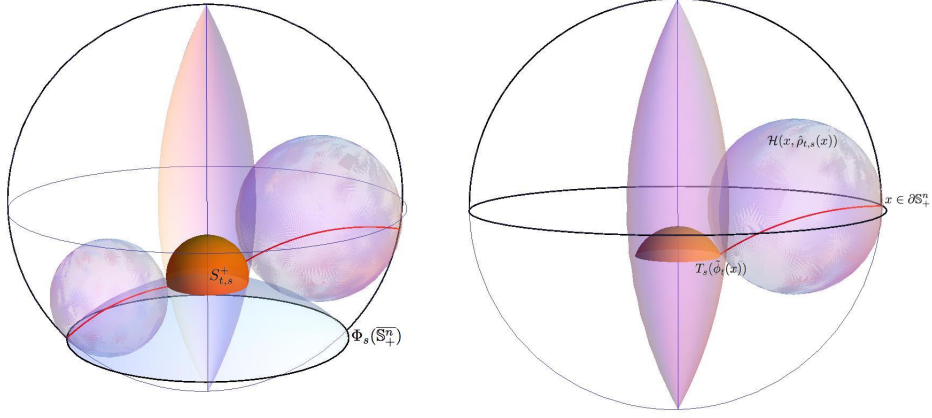


FIGURE 4. **Left:** The half-sphere $S_{t,s}^+$ translated down at distance s . Note that the image of the hyperbolic Gauss map is $G(S_{t,s}^+) = \Phi_s(\mathbb{S}_+^n)$. **Right:** In order to define $\hat{\rho}_{t,s}$ we have to restrict to a smaller cap than $S_{t,s}^+$ for any $s < 0$.

$\delta < \hat{\rho}_{t,s_0}(x) - \rho_t(x)$. Denote by β_1 the geodesic ray joining q and the point at infinity $x \in \mathbb{S}_+^n$, this arc is completely contained in the horoball determined by $\mathcal{H}(x, \hat{\rho}_{t,s_0}(x))$ and hence $\beta_1 \cap \Sigma_t = \emptyset$. Denote by β_2 the geodesic joining $\gamma(s_0)$ with the south pole $\mathbf{s} \in \mathbb{S}^n$, then $\Sigma_t \cap \beta_2 = \emptyset$, otherwise we contradict the fact that S_{t,s_0} is the first sphere of contact with Σ_t coming from infinity. Finally, denote by β_3 the geodesic arc joining $\gamma(s_0)$ and q . Consider the piecewise smooth curve $\beta = \beta_1 \cup \beta_2 \cup \beta_3$ and observe that β is homotopic to γ , moreover, $\partial \Sigma_t$ is homotopic to $\partial \mathbb{S}_+^n$, which implies that the linking number of β and $\partial \Sigma_t$ is ± 1 (depending on the orientation), that is, they must intersect. The only possibility is that they intersect in the interior of β_2 , however, this implies that Σ_t and S_{t,s_0} has a transverse intersection, contradicting that S_{t,s_0} is the first sphere of contact. Thus, $\rho_t \geq \hat{\rho}_{t,s_0}$ on \mathbb{S}_+^n . \square

Note that, since the elliptic data is homogeneous of degree one, we have that g_t satisfies

$$f(\lambda_{g_t}(p)) = f(e^{-t}\lambda_g(p)) \geq e^{-t} \text{ for all } p \in \mathbb{S}_+^n$$

and the horospherical metric of $S_{t,s}^+$ satisfies

$$f(\lambda_{\hat{g}_{t,s}}(p)) = f(e^{-t}\lambda_{g_0}(p)) = e^{-t}f(1/2, \dots, 1/2) = e^{-t} \text{ for all } p \in \mathbb{S}_+^n,$$

that is

$$f(\lambda_{g_t}(p)) \geq f(\lambda_{\hat{g}_{t,s}}(p)) \text{ for all } p \in \mathbb{S}_+^n.$$

Thus, if S_{t,s_0}^+ intersects Σ_t at an interior point, this contradicts the strong maximum principle (see Lemma 7.1 in the Appendix). Observe that we do not really need that both hyperbolic support functions are positive. To overcome this we can either dilate at the beginning with a t big enough so that $\rho_t > 0$ or translate Σ_t and S_{t,s_0} at distance $|s_0|$ using $T_{|s_0|}$. Then, the new hyperbolic support functions are positive, they coincide at some point in the interior and differ along the boundary. All these conditions follow since $T_{|s_0|}$ is an isometry.

Therefore, it remains the case that S_{t,s_0}^+ intersects Σ_t at a boundary point. Since in this case $s_0 = 0$, the argument above shows that $\rho_t \geq t$ on $\overline{\mathbb{S}_+^n}$. This inequality follows since $\rho_t \geq \hat{\rho}_{t,s}$ on $\overline{\mathbb{S}_+^n}$ for all $s < 0$, taking $s \rightarrow 0$ one can easily see that $\hat{\rho}_{t,s} \rightarrow \hat{\rho}_t := t$.

If $\partial\Sigma_t \cap P = \emptyset$, then $S_{t,s} \cap \partial\Sigma_t = \emptyset$ for all $s \in \mathbb{R}$. Hence, we can translate S_t up to the north pole until we find a first contact point with Σ_t , such point must be an interior point. However, as above, this contradicts the strong maximum principle.

Therefore, by Claim A, there exists $x \in \partial\mathbb{S}_+^n$ so that

$$\frac{\partial \hat{\rho}_t}{\partial \nu}(x) = 0,$$

hence, by the Hopf Lemma (cf. Lemma 7.2 in the Appendix), we obtain that $\rho_t \equiv t$ in $\overline{\mathbb{S}_+^n}$. Thus, $g_t = \tilde{g}_t$ and hence, $g = g_0$. \square

The same ideas work on geodesic balls in \mathbb{S}^n of radius $r < \pi/2$. However, in this situation we must impose an extra condition on the mean curvature along the boundary. Geometrically, in the previous result we compared Σ_t with the semi-sphere S_t^+ . Now, we are going to compare with a smaller spherical cap of S_t that depends on r .

First, observe that the geodesic ball $D(\mathbf{n}, r) \subset (\mathbb{S}^n, g_0)$ of radius r centered at the north pole satisfies that $\partial D(\mathbf{n}, r)$ is isometric to $\mathbb{S}^{n-1}(\sin(r))$ and the mean curvature of $\partial D(\mathbf{n}, r)$ with respect to the inward orientation is $\cot(r)$.

Second, the horospherical metric associated to the geodesic sphere $S_t \subset \mathbb{H}^{n+1}$ centered at the origin (in the Poincaré ball Model) of radius t is just the dilated metric $\tilde{g}_t = e^{2\hat{\rho}_t} g_0 = e^{2t} g_0$ and, from the representation formula (2.1), it is parametrized by

$$\tilde{\phi}_t(x) = (\cosh(t), \sinh(t)x) \text{ for all } x \in \mathbb{S}^n.$$

In particular,

$$H_{\tilde{g}_t}(x) = e^{-t} \cot(r) \text{ for all } x \in \partial D(\mathbf{n}, r).$$

Now, let P_r be the totally geodesic hyperplane in \mathbb{H}^{n+1} whose boundary at infinity coincides with the boundary of $D(\mathbf{n}, r)$, that is, $\partial_\infty P_r = \partial D(\mathbf{n}, r)$. Set $S_{r,t}^+ = \tilde{\phi}_t(\overline{D(\mathbf{n}, r)})$. Hence, with the conditions above (as we have already done) we can check that

$$\tilde{\phi}_t(x) \in \mathcal{H}(x, t) \cap P_r(\operatorname{arcsinh}(-e^{-t} \cot(r))), \text{ for all } x \in \partial D(\mathbf{n}, r)$$

and $S_{r,t}^+ \subset \overline{P_r(\operatorname{arcsinh}(-e^{-t} \cot(r)))}$ (cf. Figure 5).

Denoting by $\mathcal{B}(x, t)$ the open horoball determined by $\mathcal{H}(x, t)$ we observe that

$$\mathcal{D}(a) := P_r(\operatorname{arcsinh}(-e^{-t} \cot(r))) \setminus \bigcup_{x \in \partial D(\mathbf{n}, r)} \mathcal{B}(x, t)$$

is a closed ball in $P_r(\operatorname{arcsinh}(-e^{-t} \cot(r)))$ of radius $a > 0$ depending on r and t and centered at $q_0 = P_r(\operatorname{arcsinh}(-e^{-t} \cot(r))) \cap \gamma(\mathbb{R})$, where γ is the complete geodesic in \mathbb{H}^{n+1} joining the south and north poles. Let $\bar{a} > 0$ the unique positive number so that

$$\mathcal{C}(\bar{a}) \cap P_r(\operatorname{arcsinh}(-e^{-t} \cot(r))) = \partial \mathcal{D}(a) \subset P_r(\operatorname{arcsinh}(-e^{-t} \cot(r))),$$

where $\mathcal{C}(\bar{a})$ is the hyperbolic cylinder in \mathbb{H}^{n+1} of axis γ and radius \bar{a} , i.e., those points at distance \bar{a} from γ .

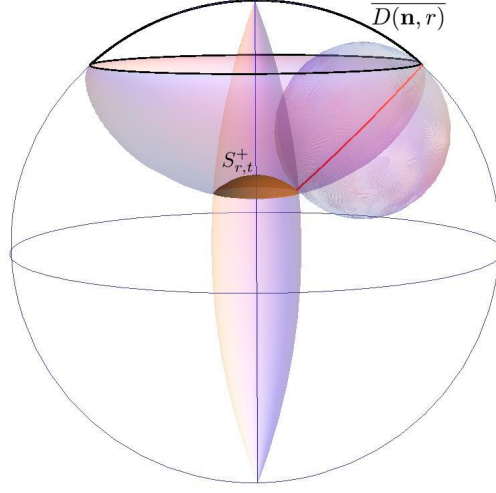


FIGURE 5. Model hypersurface $S_{r,t}^+$. In this case, the image of the hyperbolic Gauss map is $\overline{D(\mathbf{n}, r)}$, i.e., $G(S_{r,t}^+) = \overline{D(\mathbf{n}, r)}$.

The exact value of \bar{a} is not important. However it can be computed explicitly. The important observation is the following. Let $P_r(\text{arc sinh}(-e^{-t} \cot(r)))^-$ be the halfspace determined by $P_r(\text{arc sinh}(-e^{-t} \cot(r)))$ containing the south pole at its boundary at infinity, then

$$(3.4) \quad \mathcal{C}(\bar{a}) \cap P_r(\text{arc sinh}(-e^{-t} \cot(r)))^- \cap \mathcal{H}(x, t) = \emptyset \text{ for all } x \in \partial D(\mathbf{n}, r).$$

Let \hat{g}_t be the restriction of \tilde{g}_t to $\overline{D(\mathbf{n}, r)}$, i.e., $\tilde{g}_t|_{\overline{D(\mathbf{n}, r)}} = \hat{g}_t$, and $\hat{\rho}_t$ the restriction of $\tilde{\rho}_t$ to $\overline{D(\mathbf{n}, r)}$. Then, it holds

$$(3.5) \quad \hat{\rho}_t = t \text{ and } \frac{\partial \hat{\rho}_t}{\partial \nu} = 0 \text{ on } \partial D(\mathbf{n}, r),$$

where ν is the inward normal along $\partial D(\mathbf{n}, r)$ (cf. Figure 5).

After the proof of Theorem 3.2 we will explain, geometrically, the necessity on the condition for the mean curvature.

Theorem 3.2. *Let (f, Γ) be an elliptic data and let $g = e^{2\rho} g_0$, $\rho \in C^2(\overline{\mathbb{S}_+^n})$, be a supersolution to (f, Γ) in the closed hemisphere $\overline{\mathbb{S}_+^n}$, i.e.,*

$$f(\lambda(p)) \geq 1, \quad \lambda_g(p) \in \Gamma \text{ for all } p \in \mathbb{S}_+^n.$$

Assume that the boundary $\partial \mathbb{S}_+^n$ with respect to g is umbilic with mean curvature $H_g \geq \cot(r)$ and isometric to $\mathbb{S}^{n-1}(\sin r)$ for some $r \in (0, \pi/2)$, here $\mathbb{S}^{n-1}(\sin r)$ denotes the standard sphere of radius $\sin r$.

Then, there exists a conformal diffeomorphism $\Phi \in \text{Conf}(\mathbb{S}^n)$ so that $(\overline{\mathbb{S}_+^n}, \Phi^ g)$ is isometric $D(\mathbf{n}, r)$, where $D(\mathbf{n}, r)$ is the geodesic ball in \mathbb{S}^n with respect to the standard metric g_0 centered at the north pole \mathbf{n} of radius r .*

Proof of Theorem 3.2. Using Obata's Theorem in this case, up to a conformal diffeomorphism, we can assume that $g = e^{2\rho} g_0$ is defined on $\overline{D(\mathbf{n}, r)}$ and it is so that

$\rho = 0$ on $\partial D(\mathbf{n}, r)$. Moreover, the mean curvature of $\partial D(\mathbf{n}, r)$ with respect to g is given by

$$(3.6) \quad \cot(r) \leq H_g := -e^{-\rho} \frac{\partial \rho}{\partial \nu} + \cot(r) \text{ on } \partial D(\mathbf{n}, r).$$

Now, as we have done above, we fix $t > 0$ such that the eigenvalues of the Schouten tensor of $g_t = e^{2(\rho+t)} g_0$ satisfy $\lambda_i^t(x) < 1/2$, for all $x \in \overline{\mathbb{S}_+^n}$ and we denote by $\Sigma_t = \phi_t(\mathbb{S}_+^n) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ the compact embedded horospherically concave hypersurface with boundary given by the representation formula (2.1) associated to $\rho_t = \rho + t$. In particular, $\rho = t$ along $\partial D(\mathbf{n}, r)$.

As we have seen above, we have $\phi_t(x) \in \mathcal{H}(x, \rho_t(x))$, where $\mathcal{H}(x, \rho_t(x))$ is the horosphere whose point at infinity is x and distance to the origin is t . Moreover, the mean curvature $H_g(x)$ measures the equidistant where $\phi_t(x)$ is contained, that is

$$\phi_t(x) \in \mathcal{H}(x, \rho_t(x)) \cap P_r(\text{arc sinh}(-e^{-t} H_g(x))).$$

In particular, $\partial \Sigma_t \subset \overline{P_r(\text{arc sinh}(-e^{-t} \cot(r)))^-}$. Hence,

Claim: $\partial \Sigma_t$ lies outside the interior of $\mathcal{C}(\bar{a})$. Moreover, $\phi_t(x) \in \partial \Sigma_t \cap \mathcal{C}(\bar{a})$ for some $x \in \partial D(\mathbf{n}, r)$ if, and only if, $\frac{\partial \rho_t}{\partial \nu} = 0$ at $x \in \partial D(\mathbf{n}, r)$.

Proof of Claim. From (3.4), the boundary Σ_t lies outside the interior of $\mathcal{C}(\bar{a})$. Moreover, $\partial \Sigma_t$ touches $\mathcal{C}(\bar{a})$ at $\phi_t(x)$ for some $x \in \partial D(\mathbf{n}, r)$ if, and only if, $H_g(x) = \cot(r)$ and, from (3.6), this is equivalent to $\frac{\partial \rho_t}{\partial \nu} = 0$ at $x \in \partial D(\mathbf{n}, r)$. This finishes the proof of Claim. \square

Now, consider the metric $\hat{g}_t = e^{2\hat{\rho}_t}$ on $D(\mathbf{n}, r)$ defined above and satisfying (3.5). Now, we only have to compare ρ_t and $\hat{\rho}_t$ the same way we did in Theorem 3.1 and we conclude that $\rho_t \equiv \hat{\rho}_t$ on $D(\mathbf{n}, r)$. This proves the theorem. \square

The condition on the mean curvature is fundamental to ensure that $\partial \Sigma_t$ does not touch the interior of $\mathcal{C}(\bar{a})$. If, at some point $x \in \partial D(\mathbf{n}, r)$, the mean curvature were smaller than $\cot(r)$, the point $\phi_t(x)$ might be in the interior of $\mathcal{C}(\bar{a})$. Hence, when we compare Σ_t and the spherical cap, the first contact point could be an interior point of the spherical cap and a boundary point of $\partial \Sigma_t$ and hence, we can not apply the maximum principle.

Finally, we establish our main result in this section:

Theorem 3.3. *Let $p_i \in \mathbb{S}^n$ and $\epsilon_i > 0$, $i = 1, \dots, k$, be so that the closed geodesic balls $D(p_i, \epsilon_i) \subset \mathbb{S}^n$ are pairwise disjoint. Set $\Omega := \mathbb{S}^n \setminus \bigcup_{i=1}^k D(p_i, \epsilon_i)$ and let $\Lambda \subset \Omega$ be a closed subset with empty interior.*

Let (f, Γ) be an elliptic data and let $g = e^{2\rho} g_0$, $\rho \in C^2(\overline{\Omega} \setminus \Lambda)$, be a supersolution to (f, Γ) in $\Omega \setminus \Lambda$, i.e.,

$$f(\lambda(p)) \geq 1, \quad \lambda_g(p) \in \Gamma \text{ for all } p \in \Omega \setminus \Lambda.$$

Assume that g is complete in $\overline{\Omega} \setminus \Lambda$ and the Schouten tensor of g is bounded.

Assume that each boundary component $\partial D(p_i, \epsilon_i)$ with respect to g is umbilic with mean curvature $H_g \geq \cot(r)$ and isometric to $\mathbb{S}^{n-1}(\sin(r))$ for some $r \in (0, \pi/2]$, here $\mathbb{S}^{n-1}(\sin(r))$ denotes the standard sphere of radius $\sin(r)$.

Then, there exists a conformal diffeomorphism $\Phi \in \text{Conf}(\mathbb{S}^n)$ so that $(\overline{\Omega} \setminus \Lambda, \Phi^*g)$ is isometric to $\overline{D(\mathbf{n}, r)}$, where $D(\mathbf{n}, r)$ is the geodesic ball in \mathbb{S}^n with respect to the standard metric g_0 centered at the north pole \mathbf{n} of radius r .

The condition on Λ having empty interior is superfluous. Under the conditions above, following ideas contained in [2], one can prove that Λ must have empty interior.

After the proof of Theorem 3.3 we will explain the necessity of $H_g \geq 0$ when $(\partial\mathbb{S}_+^n, g)$ is isometric to \mathbb{S}^{n-1} in the case of multiple boundary components, in contrast to Theorem 3.1.

Proof of Theorem 3.3. Since $|\text{Sch}_g| < +\infty$ and g is a complete metric, following the results in [2] (see also [3]), there exists $t > 0$ such that the horospherically concave hypersurface associated

$$\Sigma_t = \phi_t(\Omega \setminus \Lambda) \subset \mathbb{H}^{n+1}$$

is properly embedded with boundary and $\partial_\infty \Sigma_t = \Lambda$. Without loss of generality we can assume that Σ_t is locally convex with respect to the canonical orientation η_t by taking t big enough.

Observe that, up to a conformal diffeomorphism $\Phi \in \text{Conf}(\mathbb{S}^n)$, we can assume that one connected component of $\partial\Omega$ is $\partial\mathbb{S}_+^n$. Consider the case where $(\partial\Omega, g)$ is isometric to (\mathbb{S}^{n-1}, g_0) . The case where $(\partial\Omega, g)$ is isometric to $(\partial\mathbb{D}(r), g_0)$ is analogous. Observe that at the beginning of Theorem 3.1 we did a conformal transformation to ensure that $\rho = 0$ along $\partial\mathbb{S}_+^n$. We can do this to ensure $\rho = 0$ along one connected component of the boundary (of course, not all of them). We assume $\Gamma_1 = \partial\mathbb{S}_+^n$ has this property. Observe that after applying this conformal diffeomorphism we can assume $\Phi(\Omega) \subset \mathbb{S}_+^n$. Now consider the half-sphere $S_t^+ \subset P^+$ as in the Theorem 3.1. We only need to prove that S_t^+ does not touch any other boundary component.

As we did in Theorem 3.1, consider the hyperbolic translation $T_s : \mathbb{H}^{n+1} \rightarrow \mathbb{H}^{n+1}$ and set $S_{t,s}^+ = T_s(S_t^+)$. Then, there exists $s_0 < 0$ so that $\Sigma_t \cap S_{t,s}^+ = \emptyset$ for all $s \geq s_0$. Then, we increase s to 0 up to the first contact point with Σ_t . If this first contact point happens either at interior points or at boundary points for $s = 0$, then Σ_t equals S_t^+ by the maximum principle as we did in Theorem 3.1.

Therefore, we only must show that the first contact point does not occur at an interior point of $S_{t,s}^+$, for some $s \in [s_0, 0]$, and a boundary point of Σ_t . Assume this happens, and let Γ_2 be the other boundary component of $\partial\Omega \setminus \Gamma_1$ that $S_{t,s}^+$ touch.

Let $p \in \mathbb{S}_+^n$ and $\epsilon > 0$, $\overline{D(p, \epsilon)} \subset \mathbb{S}_+^n$, so that $\Gamma_2 = \phi_t(\partial D(p, \epsilon))$. Let P and Q be the totally geodesic hyperplanes in \mathbb{H}^{n+1} whose boundaries at infinity are

$$\partial_\infty P = \partial\mathbb{S}_+^n \text{ and } \partial_\infty Q = \partial D(p, \epsilon).$$

Let Q^+ be the halfspace determined by Q whose boundary at infinity contains $p \in \partial_\infty Q^+$. Let $\phi_t(x) = q \in \Gamma_2 \cap S_{t,s}^+$ be a first contact point. Let η_t and $\tilde{\eta}_{t,s}$ be the canonical orientation of Σ_t and $S_{t,s}^+$ respectively. Then, $\phi_t(x) \in Q(\text{arc sinh}(e^{-t}H_g(x)))$, where $Q(\text{arc sinh}(e^{-t}H_g(x)))$ is the equidistant to Q at distance $\text{arc sinh}(e^{-t}H_g(x))$ contained in Q^+ .

Since we are assuming that the mean curvature H_g is non-negative along $\partial D(p, \epsilon)$ we have that $\eta_t(q)$ points towards $\overline{Q^+}$, it could belong to the tangent bundle of Q if $H_g(x) = 0$, $q = \phi_t(x)$. Now, since $S_{t,s}^+$ is convex with respect to the canonical orientation $\tilde{\eta}_{t,s}$, then $\tilde{\eta}_{t,s}(q)$ points towards Q^- . Since we are assuming that q is

the first contact point, the only possibility is that $\eta_t(q) = -\tilde{\eta}_{t,s}(q)$. However, if this were the case, since Σ_t and $S_{t,s}^+$ are locally convex and their tangent hyperplanes coincide, they must be (locally) in opposite sides of the tangent hyperplane, in other words, $S_{t,s}^+$ is approaching by the concave side of Σ_t , which is a contradiction. Hence, in any case, the first contact point does not occur at an interior point of $S_{t,s}^+$, for some $s \in [s_0, 0]$, and a boundary point of Σ_t .

Thus, this finishes the proof of Theorem 3.3

□

Observe that the condition $H_g \geq 0$ is essential in Theorem 3.3, in contrast to Theorem 3.1. The reason is that this condition gives us a direction of the canonical orientation η_t at the contact point. If the mean curvature at some point were negative, both η_t and $\tilde{\eta}_{t,s}$ point toward the same halfspace Q^- at the contact point q , and we can not achieve a contradiction.

4. PROOF OF THEOREM A

Now, we are ready to prove our main result. For simplicity, we divide the proof into two cases.

4.1. M is simply-connected.

Proof. First, we prove our Theorem A under the condition that M is simply-connected. In this case, there exists a developing map $\Psi : M \rightarrow \mathbb{S}^n$. Since ∂M is umbilic, and to be umbilic is a conformal invariant, the image of ∂M must be umbilic in \mathbb{S}^n . Hence, ∂M is contained in a hypersphere $\mathcal{S} \subseteq \mathbb{S}^n$. Note that, in fact, $\Psi|_{\partial M} : \partial M \rightarrow \mathcal{S}$ is a diffeomorphism. Composing it with a conformal diffeomorphism of \mathbb{S}^n , if necessary, we can assume that \mathcal{S} is the equator $\partial \mathbb{S}_+^n = \{x \in \mathbb{S}^n ; x_{n+1} = 0\}$. Now, consider the double manifold $\hat{M} = M \bigcup_{\partial M} (-M)$.

We are writing $-M$ for the second copy of M in \hat{M} in order to distinguish it from M itself. We extend Ψ to a map $\hat{\Psi} : \hat{M} \rightarrow \mathbb{S}^n$ in a natural way: we write $\Psi = (\Psi_1, \dots, \Psi_{n+1})$ and set

$$\hat{\Psi}(x) := \begin{cases} \Psi(x) & \text{if } x \in M \\ (\Psi_1(x), \dots, \Psi_{n+1}(x), -\Psi_{n+1}(x)), & \text{if } x \in -M \end{cases}$$

Then $\hat{\Psi}$ is well-defined and continuous because $\Psi_{n+1}(x) = 0$ for $x \in \partial M$. Moreover, it is a local homeomorphism. It follows that $\hat{\Psi}$ is a homeomorphism and hence Ψ is injective. Furthermore, the image is either \mathbb{S}_+^n or \mathbb{S}^n . Let $\{\mathbf{s}, \mathbf{n} = -\mathbf{s}\}$ be a pair of antipodal points. By composing Ψ with a conformal diffeomorphism of \mathbb{S}^n , we may assume that the image of Ψ is $\overline{\mathbb{S}_+^n}$.

Now, we can pushforward the metric g on M to $\overline{\mathbb{S}_+^n}$ via Ψ , $\tilde{g} = (\Psi^{-1})^*g$, and we obtain a conformal metric to standard metric on the sphere satisfying that the boundary $\partial \mathbb{S}_+^n$ with respect to \tilde{g} is umbilic with mean curvature $H_{\tilde{g}} \geq \cot(r)$ and isometric to $\mathbb{S}^{n-1}(\sin r)$ for some $r \in (0, \pi/2]$, here $\mathbb{S}^{n-1}(\sin r)$ denotes the standard sphere of radius $\sin r$. Therefore, either Theorem 3.1 if $r = \pi/2$ (in this case we do not need to assume $H_g \geq 0$) or Theorem 3.2 if $r \in (0, \pi/2)$ imply that \tilde{g} is isometric (up to a conformal diffeomorphism) to $\overline{D(\mathbf{n}, r)}$. This concludes the proof of Theorem A in the simply-connected case. □

4.2. M is not simply-connected. In this case, we will use Theorem 2.3. Then, there exists an injective conformal diffeomorphism $\Psi : M \rightarrow \Omega \setminus \Lambda$ where $\Omega = \Omega(\epsilon_i, p_i) := \mathbb{S}^n \setminus \left(\bigcup_i D(p_i, \epsilon_i) \right)$, $D(p_i, \epsilon_i)$ are geodesic balls in \mathbb{S}^n centered at p_i of radius ϵ_i with disjoint closures and Λ is a closed subset of Hausdorff dimension at most $\frac{n-2}{2}$.

Hence, as we did above, we can push forward the metric on M to $\Omega \setminus \Lambda$ as $\tilde{g} = (\Psi^{-1})^*g$, \tilde{g} is conformal to the standard metric on the sphere. This metric is complete (cf. [16, Section 2]) and its Schouten tensor is bounded, since the Schouten tensor of g is bounded in M . Moreover, the boundary conditions on g imply that each boundary component $\partial D(p_i, \epsilon_i)$ with respect to \tilde{g} is umbilic with mean curvature $H_{\tilde{g}} \geq \cot(r)$ and isometric to $\mathbb{S}^{n-1}(\sin(r))$ for some $r \in (0, \pi/2]$, here $\mathbb{S}^{n-1}(\sin(r))$ denotes the standard sphere of radius $\sin(r)$.

Therefore, Theorem 3.3 implies that there exists a conformal diffeomorphism $\Phi \in \text{Conf}(\mathbb{S}^n)$ so that $(\bar{\Omega} \setminus \Lambda, \Phi^*\tilde{g})$ is isometric to $D(\mathbf{n}, r)$, where $D(\mathbf{n}, r)$ is the geodesic ball in \mathbb{S}^n with respect to the standard metric g_0 centered at the north pole \mathbf{n} of radius r . In particular, $\Lambda = \emptyset$ and the number of connected components at the boundary is one. This implies that M is simply connected via Ψ . This concludes the proof of Theorem A.

5. RIGIDITY FOR HYPERSURFACES IN \mathbb{H}^{n+1}

Now, we will see how our results on Section 3 apply to hypersurfaces Σ in \mathbb{H}^{n+1} . We are going to establish here a simplified version of that we could, but which is geometrically more appealing.

First, we define the geometric setting. Let $P_i \subset \mathbb{H}^{n+1}$, $i = 1, \dots, m$, be pairwise disjoint totally geodesic hyperplanes and let $\mathcal{O}(m)$ be the connected component of $\mathbb{H}^{n+1} \setminus \bigcup_{i=1}^m P_i$ whose boundary is $\partial\mathcal{O}(m) = \bigcup_{i=1}^m P_i$. Fix $r \geq 0$ and denote by $P_i(r)$ the equidistant hypersurface P_i at distance r so that $P_i(r) \subset \mathbb{H}^{n+1} \setminus \mathcal{O}$. Assume that $P_i(r)$, $i = 1, \dots, m$, are pairwise disjoint and denote by $\mathcal{O}(m, r)$ the connected component of $\mathbb{H}^{n+1} \setminus \bigcup_{i=1}^m P_i(r)$ whose boundary is $\partial\mathcal{O}(m, r) = \bigcup_{i=1}^m P_i(r)$. Observe that the boundary at infinity $\bar{\Omega}(m) := \partial_\infty \mathcal{O}(m, r) \subset \mathbb{S}^n$ satisfies that $\partial\Omega(m) = \bigcup_{i=1}^m \partial D(p_i, \epsilon_i)$, for certain $p_i \in \mathbb{S}^n$ and $\epsilon_i > 0$. Moreover, we orient each P_i so that the normal N_i along P_i points into $\mathcal{O}(m, r)$. A domain $\mathcal{O}(m, r)$ in the above conditions is called a (m, r) -domain.

Second, we define how the hypersurface Σ sits into a (m, r) -domain. Let $\Sigma \subset \mathbb{H}^{n+1}$ be a properly embedded hypersurface with boundary. We say that Σ *sits into* a (m, r) -domain, denoted by $\Sigma \subset \mathcal{O}(m, r)$, if

- $\Sigma \setminus \partial\Sigma \subset \mathcal{O}(m, r)$,
- $\partial\Sigma = \bigcup_{i=1}^m \mathcal{S}_i$, where each \mathcal{S}_i is homeomorphic to \mathbb{S}^{n-1} and $\mathcal{S}_i \subset P_i(r)$,
- let $\mathcal{D}_i \subset P_i$ the domain bounded by \mathcal{S}_i in $P_i(r)$, the orientation η of Σ is the one pointing into the domain $W \subset \mathbb{H}^{n+1}$ bounded by $\Sigma \cup (\bigcup_{i=1}^m \mathcal{D}_i)$, and
- $\partial_\infty \Sigma \subset \Omega(m)$.

Third, we set the type of elliptic inequality the hypersurface will satisfy. We recall the definition of elliptic data for a hypersurface in \mathbb{H}^{n+1} (cf. [2, Section 4] and references therein). Let

$$\Gamma_n^* = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 1\}$$

and

$$\Gamma_1^* = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i > n\}.$$

Consider a symmetric function $\mathcal{W}(x_1, \dots, x_n)$ with $\mathcal{W}(1, \dots, 1) = 0$ and Γ^* an open connected component of

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : \mathcal{W}(x_1, \dots, x_n) > 0\}.$$

We say that $(\mathcal{W}, \Gamma^*, \kappa_0)$, $\kappa_0 > 0$, is an elliptic data if they satisfy

- (1) $\Gamma_n^* \subset \Gamma^* \subset \Gamma_1^*$,
- (2) \mathcal{W} is symmetric,
- (3) $\mathcal{W} > 0$ in Γ^* ,
- (4) $\mathcal{W}|_{\partial\Gamma^*} = 0$,
- (5) $\frac{\partial\mathcal{W}}{\partial x_i} > 0$ for all $i = 1, \dots, n$,
- (6) $\mathcal{W}(\kappa_0, \dots, \kappa_0) = 1$.

Then, given an elliptic data $(\mathcal{W}, \Gamma^*, \kappa_0)$ we say that an oriented hypersurface $\Sigma \subset \mathbb{H}^{n+1}$ is a *supersolution* to $(\mathcal{W}, \Gamma^*, \kappa_0)$ if

$$\mathcal{W}(k(p)) \geq 1, \quad k(p) \in \Gamma^* \text{ for all } p \in \Sigma,$$

where $k(p) := (k_1(p), \dots, k_n(p))$ is composed by the principal eigenvalues of Σ at $p \in \Sigma$ with respect to the chosen orientation.

We have already established the geometric configuration. In order to state appropriately our main result, we need to introduce some notation.

Fix $r \geq 0$ and $\kappa_0 > 1$. Let $S_p(\kappa_0)$ be the totally umbilic geodesic sphere centered at $p \in \mathbb{H}^{n+1}$ whose principal curvatures (with respect to the inward orientation) are equal to κ_0 . Let $P(r)$ be a equidistant hypersurface to a totally geodesic hyperplane P . Denote by $P(r)^+$ the convex component of $\mathbb{H}^{n+1} \setminus P$. Let $p_r \in \mathbb{H}^{n+1}$ be a point so that $S(\kappa_0, r)^+ := S_{p_r}(\kappa_0) \cap P(r)^+$ makes a constant angle $\alpha(r) = \arccos\left(-\frac{r}{\sqrt{1+r^2}}\right)$, the angle here is measure between the inward normal along the geodesic sphere and the normal along $P(r)$ pointing into the convex side.

Definition 5.1. We say that Σ is a (κ_0, r) -spherical cap if $\Sigma := S(\kappa_0, r)^+$, up to an isometry of \mathbb{H}^{n+1} .

Recall that the inradius of a closed embedded hypersurface \mathcal{S} in $P(r)$, denoted by $\text{InRad}(\mathcal{S}, P(r))$, is the radius of the biggest geodesic ball in $P(r)$ contained in the domain bounded by \mathcal{S} in $P(r)$. Then, we set $\iota(\kappa_0, r) := \text{InRad}(\partial S(\kappa_0, r)^+, P(r)) > 0$.

It is clear that (κ_0, r) -spherical caps will be the model hypersurfaces to compare with in the next result.

Theorem 5.2. Fix $m \in \mathbb{N} \cup \{0\}$ and $r \geq 0$. Consider a (m, r) -domain $\mathcal{O}(m, r)$ and let $\Sigma \subset \mathcal{O}(m, r)$ be a properly embedded hypersurface sitting on it.

Let $(\mathcal{W}, \Gamma^*, \kappa_0)$ be an elliptic data and assume that Σ is a supersolution to $(\mathcal{W}, \Gamma^*, \kappa_0)$. Assume that along the boundary Σ satisfies:

- $\langle \eta(x), N_i(x) \rangle \leq -\frac{r}{\sqrt{1+r^2}}$ for each $x \in \mathcal{S}_i$.
- $\text{InRad}(\mathcal{S}_{i_0}, P_{i_0}(r)) \geq \iota(\kappa_0, r)$ for some $i_0 \in \{1, \dots, m\}$.

Then, Σ is, up to an isometry of \mathbb{H}^{n+1} , a (κ_0, r) -spherical cap.

Proof of Theorem 5.2. The proof follows from the arguments given in Theorem 3.3. In this case, we only need to compare with the (r, κ_0) -spherical cap. \square

Remark 5.3. We can drop the embeddedness hypothesis on Theorem 5.2, as far as $\Sigma \cup (\bigcup_{i=1}^m \mathcal{D}_i)$ is Alexandrov embedded.

6. TOPONOGOV THEOREM

In this section, we proceed as Espinar-Gálvez-Mira [8] in order to define the Schouten tensor for a two-dimensional domain endowed with a metric g conformal to the standard metric g_0 on \mathbb{S}^2 . Consider $g = e^{2\rho}g_0$, where $\rho \in C^2(\Omega)$, defined on a domain $\Omega \subset \mathbb{S}^2$. In this case, we define the Schouten tensor Sch_g of g from the following relation:

$$\text{Sch}_g + \nabla^2 \rho + \frac{1}{2} \|\nabla \rho\|^2 g_0 = \text{Sch}_{g_0} + \nabla \rho \otimes \nabla \rho$$

where ∇ and ∇^2 are the gradient and the hessian with respect to the metric g_0 , respectively, and $\|\cdot\|$ denote the norm with respect of g_0 . Consider then $\lambda_g = (\lambda_1, \lambda_2)$, where λ_i , $i = 1, 2$, are the eigenvalues of the Schouten tensor given by the expression above. Note that if $f(x, y) = x + y$ then

$$f(\lambda_1, \lambda_2) = \frac{R_g}{2(n-1)} = K,$$

since $n = 2$, where K is the Gaussian curvature of $g = e^{2\rho}g_0$. Then the Liouville problem (i.e. the Yamabe problem in dimension $n = 2$) is a particular problem of more general elliptic problems for conformal metrics in \mathbb{S}^2 . Moreover, we can consider the Min-Oo conjecture for more general elliptic problems and see Toponogov's Theorem as a particular case of it. That is the subject of our next result.

Theorem 6.1. *Let (f, Γ) be an elliptic data. Let (M^2, g) be a compact surface with smooth boundary such that $f(\lambda_g) \geq 1$. Suppose the geodesic curvature k and the length L of the boundary ∂M (w.r.t. g) satisfy $k \geq c \geq 0$ and $L = \frac{2\pi}{\sqrt{1+c^2}}$ respectively. Then (M^2, g) is isometric to a disc of radius $r = \cot^{-1}(c)$ in \mathbb{S}^2 .*

Proof. Since (f, Γ) is elliptic we have that $K > 0$, where K is the Gaussian curvature of (M, g) . Hence, since the geodesic curvature k of the boundary satisfies $k \geq c \geq 0$, it follows from the Gauss-Bonnet formula that

$$2\pi\chi(M) = \int_M K dv_M + \int_{\partial\Sigma} k ds > 0,$$

where $\chi(M)$ is the Euler number of M . Therefore M is a disc. By the Riemann mapping theorem, (M^2, g) is conformally equivalent to the unit disc $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ with the flat metric ds_0^2 . Without loss of generality, we can write $g = e^{2\rho}g_0$, with $\rho \in C^2(M)$, and $M = \overline{\mathbb{S}_+^2}$, where g_0 denote the standard metric on \mathbb{S}_+^2 , since (\mathbb{D}, ds_0^2) is conformally equivalent to (\mathbb{S}_+^2, g_0) . Moreover, ρ satisfies

$$\frac{\partial \rho}{\partial \nu} = -ke^\rho \leq -ce^\rho.$$

Moreover, since $L = \frac{2\pi}{\sqrt{1+c^2}}$, we can reparametrize $\overline{\mathbb{S}_+^2}$ so that $\rho = -\ln \sqrt{1+c^2}$. Now, arguing as in the proof of Theorems 3.1 and 3.2, we obtain that (M^2, g) is isometric to a disc of radius $r = \text{arc cot}(c)$ in \mathbb{S}^2 . \square

As a direct consequence of the result above, we obtain the following version of the Toponogov Theorem.

Theorem 6.2. *Let (f, Γ) be an elliptic data. Let (M^2, g) be a closed surface such that $f(\lambda_g) \geq 1$. Assume that there exists a simple closed geodesic in M with length 2π . Then (M^2, g) is isometric to the standard sphere \mathbb{S}^2 .*

Proof. Suppose that γ is a simple closed geodesic in M with length 2π . We cut M along γ to obtain two compact surfaces with the geodesic γ as their common boundary. The result follows from applying the previous theorem to either of these two compact surfaces with boundary. \square

7. APPENDIX A: COMPARISON PRINCIPLE

In this appendix we recover some results contained in [12, 14, 15] to make this paper as self-contained as possible. Specifically, we will use [12, Lemma 6.1] and its proof, that relies in the strong maximum principle and Hopf Lemma developed in [14, 15]. We can summarize these results as follows:

Lemma 7.1 (Strong Maximum Principle). *Let (f, Γ) be an elliptic data. Let $g_i = e^{2\rho_i} g_0$, $\rho_i \in C^2(\Omega) \cap C^1(\bar{\Omega})$ for $\Omega \subset \mathbb{S}^n$, be two conformal metrics so that*

- $f(\lambda_{g_1}(p)) \geq f(\lambda_{g_2}(p))$, $\lambda_{g_i}(p) \in \Gamma$, $i = 1, 2$, for all $p \in \Omega$,
- $\rho_1, \rho_2 > 0$.

If $\rho_1 - \rho_2 > 0$ on $\partial\Omega$ then $\rho_1 - \rho_2 > 0$ on Ω .

And

Lemma 7.2 (Hopf Lemma). *Let (f, Γ) be an elliptic data. Let $g_i = e^{2\rho_i} g_0$, $\rho_i \in C^2(\Omega) \cap C^1(\bar{\Omega})$ for $\Omega \subset \mathbb{S}^n$, be two conformal metrics so that*

- $f(\lambda_{g_1}(p)) \geq f(\lambda_{g_2}(p))$, $\lambda_{g_i}(p) \in \Gamma$, $i = 1, 2$, for all $p \in \Omega$,
- $\rho_1 \geq \rho_2 > 0$.

If $\frac{\partial}{\partial\eta}(\rho_1 - \rho_2) \leq 0$ at $p \in \partial\Omega$ then $\rho_1 = \rho_2$ on Ω .

We should say that the results in [12] do not need that f is homogeneous of degree one. Also, in [12], the authors assumed $f \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$, but it suffices $f \in C^1(\Gamma) \cap C^0(\bar{\Gamma})$.

REFERENCES

- [1] D.P. Abanto, J.M. Espinar, *Escobar's Type Theorems for elliptic fully nonlinear degenerate equations*. Preprint, arXiv:1606.07530.
- [2] V. Bonini, J. M. Espinar, J. Qing, *Hypersurfaces in the Hyperbolic Space with support function*, Adv. Math., **280** (2015), 506–548.
- [3] V. Bonini, J. Qing, J. Zhu, *Weakly Horospherically Convex Hypersurfaces in Hyperbolic Space*. Preprint, arXiv:1611.06421.
- [4] S. Brendle, F. C. Marques, A. Neves, *Deformations of the hemisphere that increase scalar curvature*, Invent. Math., **185** (2011) no. 1, 175–197.
- [5] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian*, Acta Math., **155** (1985) no. 3–4, 261–301.
- [6] M. P. Cavalcante, J. M. Espinar, *Uniqueness Theorems for fully nonlinear conformal equations on subdomains of the sphere*. Preprint, arXiv:1505.00733.
- [7] J.M. Espinar, *Invariant Conformal Metrics on \mathbb{S}^n* , Trans. Amer. Math. Soc., **363** (2011) no. 11, 5649–5662.

- [8] J.M. Espinar, J. A. Gálvez, P. Mira, *Hypersurfaces in \mathbb{H}^{n+1} and conformally invariant equations: the generalized Christoffel and Nirenberg problems*, J. Eur. Math. Soc. **11** (2009) no. 4, 903–939.
- [9] K. Fukui, T. Nakamura, *A topological property of Lipschitz mappings*, Topology and its Applications, **148** (2005), 143–152.
- [10] F. Hang, X. Wang, *Rigidity and Non-rigidity Results on the Sphere*, Comm. Anal. Geom., **14** (2006) no. 1, 91–106.
- [11] F. Hang, X. Wang, *Rigidity theorems for compact manifolds with boundary and positive Ricci curvature*, J. Geom. Analysis, **19** (2009) no. 3, 628–642.
- [12] Q. Jin, A. Li, Y.Y. Li, *Estimates and existence results for fully nonlinear Yamabe problems on manifolds with boundary*, Calc. Var. PDEs, **28** (2007), 509–543.
- [13] N.H. Kuiper, *On conformally-flat spaces in the large*. Ann. of Math. (2) 50, (1949). 916–924.
- [14] A. Li, Y.Y. Li, *On some conformally invariant fully nonlinear equations*. Comm. Pure Appl. Math., **56** (2003), 1416–1464.
- [15] A. Li, Y.Y. Li, *On some conformally invariant fully nonlinear equations. II. Liouville, Harnack and Yamabe*, Acta Math., **195** (2005), 117–154.
- [16] Y.Y. Li, L. Nguyen, *A fully nonlinear version of the Yamabe problem on locally conformally flat manifolds with umbilic boundary*, Adv. Math., **251** (2014), 87–110.
- [17] M. Min-Oo, *Scalar curvature rigidity of certain symmetric spaces*, Geometry, topology, and dynamics (Montreal, PQ, 1995), CRM Proc. Lecture Notes, vol. 15, Amer. Math. Soc., Providence, RI, 1998, pp. 127–136.
- [18] J.G. Ratcliffe, *Foundations of hyperbolic manifolds*. Second edition. Graduate Texts in Mathematics, 149. Springer, New York, 2006.
- [19] R. Schoen. S.T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys., **65** (1979) no. 1, 45–76.
- [20] R. Schoen. S.T. Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys., **79** (1981) no. 2, 231–260.
- [21] R. Schoen, S.T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988) no. 1, 47–71.
- [22] R. Schoen, S.T. Yau, *Lectures on differential geometry*. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. Translated from the Chinese by Ding and S. Y. Cheng. Preface translated from the Chinese by Kaising Tso. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.
- [23] F. M. Spiegel, *Scalar curvature rigidity for locally conformally flat manifolds with boundary*. Preprint, arXiv:1511.06270v2.
- [24] V. A. Toponogov, *Evaluation of the length of a closed geodesic on a convex surface*, Dokl. Akad. Nauk SSSR, **124** (1959), 282–284.

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE MINAS GERAIS, BELO HORIZONTE-BRAZIL

E-mail address: ezequiel@mat.ufmg.br

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE ALAGOAS, MACEIÓ-BRAZIL

E-mail address: marcos@pos.mat.ufal.br

INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA, RIO DE JANEIRO - BRAZIL

E-mail address: jespinar@impa.br